Appendix A. Linear stability analysis

The system in question is

$$\begin{cases} \dfrac{\partial F}{\partial \tau} = -\lambda \dfrac{\partial H}{\partial \lambda} \dfrac{1}{H(\lambda,\tau)} - F(\lambda,\tau) \\ -\dfrac{\partial^2 H}{\partial \tau \partial \lambda} = P(\lambda) F(\lambda,\tau) + C\lambda \dfrac{\partial H}{\partial \lambda} \end{cases} \qquad \text{(A.1)}$$

Put $H(\lambda,\tau) = h(\lambda) + \tilde{h}(\lambda,\tau)$ and $F(\lambda,\tau) = f(\lambda) + \tilde{f}(\lambda,\tau)$ where (0, 1) and (*f, h*) are steady states.

Expanding (A.1) about the steady state gives

$$\begin{cases} \dfrac{\partial \tilde{f}}{\partial \tau} = -\lambda \dfrac{\partial}{\partial \lambda}\Big(h(\lambda) + \tilde{h}(\lambda,\tau) \Big) \dfrac{1}{h(\lambda) + \tilde{h}(\lambda)} - \Big(f(\lambda) + \tilde{f}(\lambda,\tau) \Big) \\ -\dfrac{\partial^2 \tilde{h}}{\partial \tau \partial \lambda} = P\Big(f(\lambda) + \tilde{f}(\lambda,\tau) \Big) + C\lambda \dfrac{\partial}{\partial \lambda}\Big(h(\lambda) + \tilde{h}(\lambda,\tau) \Big) \end{cases} \qquad \text{(A.2)}$$

A.1 Steady state (0, 1)

In this case, (A.2) becomes

$$\frac{\partial \tilde{f}}{\partial \tau} = -\lambda \frac{\partial \tilde{h}(\lambda,\tau)}{\partial \lambda} \frac{1}{1 + \tilde{h}(\lambda,\tau)} - \tilde{f}(\lambda,\tau)$$

$$-\frac{\partial^2 \tilde{h}}{\partial \lambda \partial \tau} = P \tilde{f}(\lambda,\tau) + C\lambda \frac{\partial \tilde{h}}{\partial \lambda}$$

This can be expressed as

$$\Big[1 + \tilde{h}(\lambda,\tau) \Big] \frac{\partial \tilde{f}}{\partial \tau} = -\lambda \frac{\partial \tilde{h}(\lambda,\tau)}{\partial \lambda} - \Big(1 + \tilde{h}(\lambda,\tau) \Big) \tilde{f}(\lambda,\tau)$$

$$-\frac{\partial^2 \overset{\square}{h}}{\partial\tau\partial\lambda} = P\overset{\square}{f}\left(\lambda,\tau\right) + C\lambda\frac{\partial \overset{\square}{h}}{\partial\lambda}$$

and retaining terms to $O(\overset{\square}{h})$ and $O(\overset{\square}{f})$

$$\begin{cases} \dfrac{\partial \overset{\square}{f}}{\partial\tau} = -\lambda\dfrac{\partial \overset{\square}{h}}{\partial\lambda} - f\left(\overset{\square}{\lambda},\tau\right) \\ -\dfrac{\partial^2 \overset{\square}{h}}{\partial\tau\lambda} = P\overset{\square}{f}\left(\lambda,\tau\right) + C\lambda\dfrac{\partial \overset{\square}{h}}{\partial\lambda} \end{cases} \qquad \text{(A.3)}$$

Substituting $h^{'} = \frac{\partial \overset{\square}{h}}{\partial\lambda}$ into (A.3)

$$\frac{\partial \overset{\square}{f}}{\partial\tau} = -\lambda h^{'} - \overset{\square}{f}\left(\lambda,\tau\right) \qquad \text{(A.4)}$$

$$\frac{\partial h^{'}}{\partial\tau} = -P\overset{\square}{f}\left(\lambda,\tau\right) - C\lambda h^{'} \qquad \text{(A.5)}$$

Differentiating (A.5) with respect to time,

$$\frac{\partial^2 \overset{\square}{h}}{\partial\tau^2} = -P\frac{\partial \overset{\square}{f}}{\partial\tau} - C\lambda\frac{\partial h^{'}}{\partial\tau}$$

and substituting (A.4),

$$\frac{\partial^2 h^{'}}{\partial\tau^2} = -P\left[-\lambda h^{'} - \overset{\square}{f}\left(\lambda,\tau\right)\right] - C\lambda\frac{\partial h^{'}}{\partial\tau}$$

or

$$\frac{\partial^2 h^{'}}{\partial\tau^2} + P\left[-\lambda h^{'} - \overset{\square}{f}\left(\lambda,\tau\right)\right] + C\lambda\frac{\partial h^{'}}{\partial\tau} = 0 \qquad \text{(A.6)}$$

where

$$-\overset{\square}{f} = \frac{1}{P}\frac{\partial h^{'}}{\partial\tau} + \frac{C}{P}\lambda h^{'}$$

then (A.6) becomes

$$\frac{\partial^2 h'}{\partial \tau^2}+P\left[-\lambda h'+\frac{1}{P}\frac{\partial h'}{\partial \tau}+\frac{C}{P}\lambda h'\right]+C\lambda\frac{\partial h'}{\partial \tau}=0$$

or

$$\frac{\partial^2 h'}{\partial \tau^2}-\lambda h' P+C\lambda h'+\frac{\partial h'}{\partial \tau}+C\lambda\frac{\partial h'}{\partial \tau}=0$$

and therefore,

$$\frac{\partial^2 h'}{\partial \tau^2}+\left(1+C\lambda\right)\frac{\partial h'}{\partial \tau}+\left(C-P\right)\lambda h'=0 \tag{A.7}$$

Similarly, differentiating (A.4) with respect to time

$$\frac{\partial^2 \overset{\square}{f}}{\partial \tau^2}=-\lambda\frac{\partial h'}{\partial \tau}-\frac{\partial \overset{\square}{f}}{\partial \tau}$$

and substituting (A.5) gives

$$\frac{\partial^2 \overset{\square}{f}}{\partial \tau^2}=-\lambda\left[-P\overset{\square}{f}-C\lambda h'\right]-\frac{\partial \overset{\square}{f}}{\partial \tau}$$

From (A.4)

$$-\lambda h'=\frac{\partial \overset{\square}{f}}{\partial \tau}+\overset{\square}{f}\left(\lambda,\tau\right)$$

Therefore,

$$\frac{\partial^2 f}{\partial \tau^2}=-\lambda\left[-P\overset{\square}{f}+C\frac{\partial \overset{\square}{f}}{\partial \tau}+C\overset{\square}{f}\left(\lambda,\tau\right)\right]-\frac{\partial \overset{\square}{f}}{\partial \tau}$$

or

$$\frac{\partial^2 \overset{\square}{f}}{\partial \tau^2}+\lambda\overset{\square}{f}\left(C-P\right)+\frac{\partial \overset{\square}{f}}{\partial \tau}+\lambda C\frac{\partial \overset{\square}{f}}{\partial \tau}=0$$

and therefore,

$$\frac{\partial^2 \overset{\square}{f}}{\partial \tau^2}+\left(1+C\lambda\right)\frac{\partial \overset{\square}{f}}{\partial \tau}+\left(C-P\right)\lambda\overset{\square}{f}=0 \tag{A.8}$$

The characteristic polynomial for (A.7) and (A.8) is

$$m^2+(1+C\lambda)m+\lambda(C-P)=0$$

When
with roots given by

$$m=\frac{1}{2}\left[-(1+C\lambda)\pm\sqrt{(1+C\lambda)^2-4\lambda(C-P)}\right]$$

Now the radical is

$$\begin{aligned}(1+C\lambda)^2-4\lambda(C-P)&\\ &=1+2C\lambda+C^2\lambda^2-4C\lambda+4\lambda P\\ &=1-2C\lambda+C^2\lambda^2+4\lambda P\end{aligned}$$

With with $=(1-C\lambda)^2+4\lambda P>0$

Therefore, roots are real.
Roots are negative definite iff

$1+C\lambda>0$ (true)

and $C-P>0$

Therefore, the system is linearly stable (exponentially) provided $P/C < 1$.
Therefore, (0, 1) is a stable node if $P/C < 1$. It is sufficient to show that f=0 is stable, since if there are no flowers, the seeds cannot be produced.

Note that even is we prove stability for h', *e.g.* suppose $h'=e^{-m\tau}g(\lambda)$ then $\tilde{h}=\int h'd\lambda=e^{-m\tau}\int g(\lambda)d\lambda+q(\lambda)$ where $q(\lambda)$ is arbitrary so $\tilde{h}$ does not necessarily tend to zero if h' does. However, since f=0 implies that s=0 we must have $q(\lambda)$=0.

A.2 Steady state *(f, h)*

In this case P/C is a function of λ, otherwise *f=0*.
From equation (A.2),

$$\left(h+\overset{\square}{h}\right)\frac{\partial \overset{\square}{f}}{\partial\tau}=-\lambda\frac{\partial}{\partial\lambda}\left(h+\overset{\square}{h}\right)-\left(h+\overset{\square}{h}\right)\left(f+\overset{\square}{f}\right)$$

and

$$-\frac{\partial^2 \overset{\square}{h}}{\partial\tau\partial\lambda}=P\left(f+\overset{\square}{f}\right)+C\lambda\frac{\partial}{\partial\lambda}\left(h+\overset{\square}{h}\right)$$

Linearise the system to get

$$h\frac{\partial \overset{\square}{f}}{\partial\tau}=-\lambda\frac{\partial}{\partial\lambda}\left(h+\overset{\square}{h}\right)-hf-\overset{\square}{h}f-\overset{\square}{f}h \tag{A.9}$$

$$-\frac{\partial^2 \overset{\square}{h}}{\partial\tau\partial\lambda}=P\left(f+\overset{\square}{f}\right)+C\lambda\frac{\partial}{\partial\lambda}\left(h+\overset{\square}{h}\right) \tag{A.10}$$

Dividing (A.10) by *C* gives

$$-\frac{1}{C}\frac{\partial^2 \overset{\square}{h}}{\partial\tau\partial\lambda}=\frac{P}{C}\left(f+\overset{\square}{f}\right)+\lambda\frac{\partial}{\partial\lambda}\left(h+\overset{\square}{h}\right)$$

and setting

$$p(\lambda)=\frac{P}{C},$$

$$-\frac{1}{C}\frac{\partial^2 \overset{\square}{h}}{\partial\tau\partial\lambda}=p\left(f+\overset{\square}{f}\right)+\lambda\frac{\partial}{\partial\lambda}\left(h+\overset{\square}{h}\right) \tag{A.11}$$

In the steady state, h=p and $f=-\frac{\lambda}{p}\frac{dp}{d\lambda}$.

Using these results in (A.9) and (A.11) gives

$$\begin{cases} p\frac{\partial \overset{\square}{f}}{\partial\tau}=-\lambda\frac{\partial}{\partial\lambda}\left(p+\overset{\square}{h}\right)-p\left(-\frac{\lambda}{p}\frac{dp}{d\lambda}\right)-\overset{\square}{h}\left(-\frac{\lambda}{p}\frac{dp}{d\lambda}\right)-\overset{\square}{f}p \\ -\frac{1}{C}\frac{\partial^2 \overset{\square}{h}}{\partial\tau\partial\lambda}=p\left(-\frac{\lambda}{p}\frac{dp}{d\lambda}+\overset{\square}{f}\right)+\lambda\frac{\partial}{\partial\lambda}\left(p+\overset{\square}{h}\right) \end{cases}$$

Simplifying

$$\begin{cases} p\frac{\partial \overset{\square}{f}}{\partial\tau}=-\lambda\left(\frac{dp}{d\lambda}+\frac{\partial \overset{\square}{h}}{\partial\lambda}\right)+\lambda\frac{dp}{d\lambda}+\frac{\lambda}{p}\frac{dp}{d\lambda}\overset{\square}{h}-p\overset{\square}{f} \\ -\frac{1}{C}\frac{\partial^2 \overset{\square}{h}}{\partial\tau\partial\lambda}=-\lambda\frac{dp}{d\lambda}+p\overset{\square}{f}+\lambda\frac{dp}{d\lambda}+\lambda\frac{\partial \overset{\square}{h}}{\partial\lambda} \end{cases}$$

$$\begin{cases} p\dfrac{\partial \overset{\square}{f}}{\partial\tau} = -\lambda\dfrac{\partial \overset{\square}{h}}{\partial\lambda} + \dfrac{\lambda}{p}\dfrac{dp}{d\lambda}\overset{\square}{h} - p\overset{\square}{f} \\ -\dfrac{1}{C}\dfrac{\partial^2 \overset{\square}{h}}{\partial\tau\partial\lambda} = p\overset{\square}{f} + \lambda\dfrac{\partial \overset{\square}{h}}{\partial\lambda} \end{cases} \qquad (A.12)$$

Substituting for $p\overset{\square}{f} + \lambda\dfrac{\partial \overset{\square}{h}}{\partial\lambda}$ in the first equation of (A.12) using the second equation gives

$$p\frac{\partial \overset{\square}{f}}{\partial\tau} = \frac{1}{C}\frac{\partial^2 \overset{\square}{h}}{\partial\tau\partial\lambda} + \frac{\lambda}{p}\frac{dp}{d\lambda}\overset{\square}{h} \qquad (A.13)$$

Now, the second equation in (A.12) can be rewritten as

$$p\overset{\square}{f} = -\frac{1}{C}\frac{\partial^2 \overset{\square}{h}}{\partial\tau\partial\lambda} - \lambda\frac{\partial \overset{\square}{h}}{\partial\lambda}$$

and differentiating with respect to τ gives

$$p\frac{\partial \overset{\square}{f}}{\partial\tau} = -\frac{1}{C}\frac{\partial^3 h}{\partial^2\tau\partial\lambda} - \lambda\frac{\partial^2 \overset{\square}{h}}{\partial\tau\partial\lambda}$$

Using this in (A.13) gives

$$\frac{1}{C}\frac{\partial^3 \overset{\square}{h}}{\partial^2\tau\partial\lambda} + \lambda\frac{\partial^2 \overset{\square}{h}}{\partial\lambda\partial\tau} + \frac{1}{C}\frac{\partial^2 \overset{\square}{h}}{\partial\tau\partial\lambda} + \frac{\lambda}{p}\frac{dp}{d\lambda}\overset{\square}{h} = 0$$

First equation in (A.12) can be rewritten as

$$\frac{\partial \overset{\square}{f}}{\partial\tau} = -\frac{\lambda}{p}\frac{\partial \overset{\square}{h}}{\partial\lambda} + \frac{\lambda}{p^2}\frac{dp}{d\lambda}\overset{\square}{h} - \overset{\square}{f}$$

i.e.

$$\frac{\partial \overset{\square}{f}}{\partial\tau} = -\lambda\left[\frac{1}{p}\frac{\partial \overset{\square}{h}}{\partial\lambda} - \frac{1}{p^2}\frac{dp}{d\lambda}\overset{\square}{h}\right] - \overset{\square}{f}$$

i.e.

$$\frac{\partial \overset{\square}{f}}{\partial\tau} = -\lambda\frac{\partial}{\partial\lambda}\left[\frac{\overset{\square}{h}}{p}\right] - \overset{\square}{f}$$

and finally,

$$\frac{\partial \overset{\square}{f}}{\partial\tau} + \overset{\square}{f} = -\lambda\frac{\partial}{\partial\lambda}\left[\frac{\overset{\square}{h}}{p}\right] \qquad (A.14)$$

Look for separable solutions to (A.14) of the form $\overset{\square}{h} = e^{\delta\tau} L(\lambda)$, then (A.14) can be rewritten as

$$\frac{\partial \overset{\square}{f}}{\partial \tau} + \overset{\square}{f} = -\lambda \frac{\partial}{\partial \lambda}\left[\frac{e^{\delta\tau} L}{p}\right]$$

i.e.

$$\frac{\partial \overset{\square}{f}}{\partial \tau} + \overset{\square}{f} = u(\lambda) e^{\delta\tau} \qquad \text{(A.15)}$$

where

$$u(\lambda) = -\lambda \frac{d}{d\lambda}\left(\frac{L}{p}\right)$$

with solution

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for *v* some arbitrary function of λ and $\delta \neq -1$.

If $\delta = -1$ then (A.15) is

$$\frac{\partial \overset{\square}{f}}{\partial \tau} + \overset{\square}{f} = u(\lambda) e^{-\tau}$$

with solution

$$\overset{\square}{f} = \tau e^{-\tau} u(\lambda) + v(\lambda) e^{-\tau}$$

i.e. $\overset{\square}{f}$ and $\overset{\square}{h}$ $\rightarrow$ 0 when $\tau \rightarrow \infty$ which implies stability of $\overset{\square}{f}$ and $\overset{\square}{h}$ for $\delta = -1$.

Now, consider the consequences for *F* and *H*.

Note that

$$H = p + \overset{\square}{h} = p + e^{-\tau} L(\lambda) = 1 + \int_{\lambda}^{1} s(l,\tau)\, dl$$

Differentiating with respect to λ

$$\frac{dp}{d\lambda} + e^{-\tau} \frac{dL}{d\lambda} = -s(\lambda,\tau)$$

and in the limit $\tau \rightarrow \infty$, we have

$$\frac{dp}{d\lambda} = -s(\lambda,\tau_\infty)$$

Since $s(\lambda,\tau_\infty) > 0 \forall \lambda \Rightarrow \dfrac{dp}{d\lambda} < 0$. (A.16)

If $\delta \neq 1$, then

$$\overset{\square}{f} = \frac{e^{\delta\tau}}{\delta+1} u(\lambda) + v(\lambda) e^{-\tau}$$

where $u(\lambda) = -\lambda \frac{d}{d\lambda}\left(\frac{L}{p}\right)$

and

$$\overset{\square}{h} = e^{\delta\tau} L(\lambda)$$

with arbitrary $v(\lambda)$.

Now, the second equation of (A.12) can be written as

$$-\frac{1}{C}\delta e^{\delta\tau}\frac{dL}{d\lambda} = -p\lambda\frac{d}{d\lambda}\left(\frac{L}{p}\right)\frac{e^{\delta\tau}}{\delta+1} + pv(\lambda)e^{\delta\tau} + \lambda e^{\delta\tau}\frac{dL}{d\lambda}$$

$$\frac{1}{C}\delta\frac{dL}{d\lambda} - p\lambda\frac{1}{\delta+1}\frac{d}{d\lambda}\left(\frac{L}{p}\right) + pv(\lambda) + \lambda\frac{dL}{d\lambda} = 0$$

$$\left(\frac{1}{C}\delta+\lambda\right)\frac{dL}{d\lambda} + pv(\lambda) - \frac{p\lambda}{\delta+1}\left[\frac{p\frac{dL}{d\lambda} - L\frac{dp}{d\lambda}}{p^2}\right] = 0$$

$$\left(\frac{\delta}{C}+\lambda\right)(\delta+1)\frac{dL}{d\lambda} + (\delta+1)pv(\lambda) - \frac{\lambda}{p}\left[p\frac{dL}{d\lambda} - L\frac{dp}{d\lambda}\right] = 0$$

$$\left[\left(\frac{\delta}{C}+\lambda\right)(\delta+1) - \lambda\right]\frac{dL}{d\lambda} + (\delta+1)pv(\lambda) + \frac{\lambda}{p}L\frac{dp}{d\lambda} = 0$$

$$\left[\frac{\delta^2}{C} + \delta\lambda + \frac{\delta}{C} + \lambda - \lambda\right]\frac{dL}{d\lambda} + (\delta+1)pv(\lambda) + \frac{\lambda}{p}L\frac{dp}{d\lambda} = 0$$

$$\left[\frac{1}{C}\delta^2 + \lambda\delta + \frac{1}{C}\delta\right]\frac{dL}{d\lambda} + (\delta+1)pv(\lambda) + \frac{\lambda}{p}L\frac{dp}{d\lambda} = 0$$

$$\left(\frac{1}{C}\delta^2 + \left(\lambda + \frac{1}{C}\right)\delta\right)\frac{1}{L}\frac{dL}{d\lambda} + \frac{(\delta+1)pv(\lambda)}{L(\lambda)} + \frac{\lambda}{p}\frac{dp}{d\lambda} = 0 \qquad \text{(A.17)}$$

Return to consider possible forms of $v(\lambda)$

$$\overset{\square}{f} = \frac{e^{\delta\tau}}{\delta+1} u(\lambda) + v(\lambda) e^{-\tau}$$

where $\delta \neq -1$

Now, since $\delta \neq -1$ => $\overset{\square}{f} = \frac{e^{\delta\tau}}{\delta+1} u(\lambda)\left[1 + (\delta+1)\frac{v(\lambda)}{u(\lambda)} e^{-(1+\delta)\tau}\right]$

If $\delta > -1$ and $\tau \to \infty$, $\overset{\square}{f} = \frac{e^{\delta\tau}}{\delta + 1} u(\lambda)$

If $\delta < -1$ => $\overset{\square}{f} = v(\lambda) e^{-\tau}$ as $\tau \to \infty$ and this is stable for $\frac{dp}{d\lambda} < 0$ from equation (A.16).

Therefore, $\delta \le -1$ gives rise to stable solutions.

Now consider $\delta > -1$.

Then $\overset{\square}{f} = \frac{e^{\delta\tau}}{\delta + 1} u(\lambda)$ for large τ.

Substituting this into the second equation of (A.12) gives a simple form of equation (A.17), *i.e.*

$$\delta\left[\frac{\delta}{C} + \lambda + \frac{1}{C}\right] = \left. \frac{-\lambda}{p}\frac{dp}{d\lambda} \middle/ \frac{1}{L}\frac{dL}{d\lambda} \right. = w(\lambda)$$

but $L(\lambda)$ is still arbitrary since $w(\lambda)$ can be positive or negative.

If $w(\lambda) > 0 \Rightarrow \delta > 0$ or $\delta < -c\lambda - 1$.

But $\delta > -1$ and therefore, $w(\lambda) > 0 \Rightarrow \delta > 0$.

If $w(\lambda) < 0 \Rightarrow -(c\lambda + 1) < \delta < 0$.

But $\delta < -1$ and therefore, $-1 < \delta < 0$.

Therefore, dependently on the form of $L(\lambda)$, it is possible that $\delta > 0$, and therefore, the system may not be stable. Therefore, $L(\lambda)$ needs to be constrained with additional boundary conditions. At present, it is not clear what these should be.